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# EQUITABLE COLORING OF PRISMS AND THE GENERALIZED PETERSEN GRAPHS

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## ABSTRACT

Gallian in 2007 gave the construction of the prism $Y_m^n$ by considering the cartesian product of the cycle $C_m$ and the path $P_n$. The generalized Petersen graph was introduced in 1950 by Coxeter. The generalized Petersen graphs are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. A graph G is said to be equitable $k$-coloring if the vertex set $V(G)$ is partitioned into disjoint independent sets so that the size of each partition differs at most one by the rest of the partitions. In this paper, we discussed the equitable coloring of the prisms and obtained the result that its chromatic number always lies between 2 and 3. We have also discussed the equitable coloring of the generalized Petersen graphs $P(m,n), m \geq 2n+1, n > 1$.

**AMS Classification Code:** 05C15

**KEYWORDS:** Prism Graph, Petersen Graph, Equitable Coloring, Color Class, Chromatic Number of Equitable Coloring

## INTRODUCTION

Mayer [1] introduced the equitable chromatic number of a graph and Lih [2,3] have elaborately discussed about the equitable coloring of graphs and in particular about bipartite graphs and trees. Dorothee [4] has discussed about the equitable coloring of complete multipartite graphs. A prism $Y_m^n$ is a simple graph and is obtained as the cartisian product of the cycle $C_m$ and the path $P_n$. The prism $Y_m^n$ has $mn$ vertices and $m(2n-1)$ edges. Coxeter [5] introduced the generalization of Petersen graphs. An equitable coloring is an assignment of colors to the vertices of a graph in such a way that no two adjacent vertices have the same color and the number of vertices in any two color classes differ at most by one. Application of equitable coloring is found in scheduling and timetabling problems.

**Definition 1**

Graph coloring is the coloring the vertices of a graph with the minimum number of colors without any two adjacent vertices having the same color.

**Definition 2**

In vertex coloring of a graph, the set of vertices of same color are said to be in that color class. In k-coloring of a graph, there are k color classes. They are represented by $C[1], C[2], \ldots$ , where 1,2, ... denote the colors.

**Definition 3**

A graph G is said to be equitable k-colorable if its vertex set V can be partitioned into k subsets $V_1, V_2, \ldots, V_k$, such that each $V_i$ is an independent set and the condition $\left||V_i| - |V_j|\right| \leq 1$ holds for all $1 \leq i,j \leq k$. The smallest integer k for which G is equitable k-coloring is known as the equitable chromatic number of G and is denoted by $\chi_= G$.

**Definition 4**

The cartisian product of the cycle $C_m$ and the path $P_n$ is said to be the prism graph and is denoted by $Y_m^n$.

## EQUITABLE COLORING OF PRISM GRAPHS

**Theorem 1**

The prism graph $Y_m^n$ admits equitable coloring and its chromatic number lies between 2 and 3.

**Proof:** For the prism graph $Y_m^n$, we consider the cartesian product of $C_m$ and $P_n$. In $Y_m^n$, the subscript m stands for the order of the cycle and the superscript n stands for the number of vertices in the path $P_n$. We represent the vertices of the first cycle by $v_1^1, v_2^1, v_3^1, \dots v_m^1$, the vertices of the second cycle by $v_1^2, v_2^2, v_3^2, \dots v_m^2$ and so on. The spokes $v_i^j v_i^{j+1}$, $1 \le i \le m$ and $1 \le j \le n$ are as shown in the figure 1.

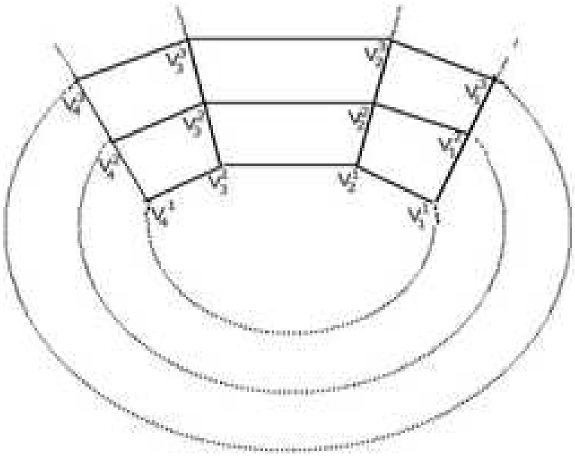

**Figure 1**

The function '$f$' is defined as the coloring from the vertices of the prism $Y_m^n$ to the set of colors $\{1,2,3,\dots\}$ as follows:

**Case (i):** Let m be even. Define for $1 \le i \le m$ and $1 \le j \le n$

$$f\left(v_i^j\right) = \begin{cases} 1, if\ i+j\ is\ even \\ 2, if\ i+j\ is\ odd \end{cases} \qquad (1)$$

Here we get $|C[1]| = |C[2]|$. The prism $Y_m^n$ admits equitable coloring with this type of coloring.

Hence $\chi_{=}(Y_m^n) = 2$ if m is even.

**Case (ii):** Let m be odd and $m \not\equiv 1(mod3)$, we define the function $f$, $1 \le i \le m$ and $1 \le j \le n$ as follows:

For $j \equiv 1(mod3)$

$$f\left(v_i^j\right) = \begin{cases} 1, if\ i \equiv 1(mod3) \\ 2, if\ i \equiv 2(mod3) \\ 3, if\ i \equiv 0(mod3) \end{cases} \qquad (2)$$

For $j \equiv 2(mod3)$

$$f(v_i^j) = \begin{cases} 2, if\ i \equiv 1(mod3) \\ 3, if\ i \equiv 2(mod3) \\ 1, if\ i \equiv 0(mod3) \end{cases} \tag{3}$$

For $j \equiv 0(mod3)$

$$f(v_i^j) = \begin{cases} 3, if\ i \equiv 1(mod3) \\ 1, if\ i \equiv 2(mod3) \\ 2, if\ i \equiv 0(mod3) \end{cases} \tag{4}$$

The color classes $C[1], C[2]$ and $C[3]$ satisfy the conditions $\big||C[i]| - |C[j]|\big| \leq 1, 1 \leq i \leq 3$ and $1 \leq j \leq 3$. The prism $Y_m^n$ admits equitable coloring with this type of coloring.

Hence $\chi_=(Y_m^n) = 3$ if m is odd and $m \not\equiv 1(mod3)$

**Case (iii):** Let m be odd and $m \equiv 1(mod3)$, we define the function $f$, $1 \leq i < m$ and $1 \leq j \leq n$ as follows:

For $j \equiv 1(mod3)$

$$f(v_i^j) = \begin{cases} 1, if\ i \equiv 1(mod3) \\ 2, if\ i \equiv 2(mod3) \\ 3, if\ i \equiv 0(mod3) \end{cases} \tag{5}$$

For $j \equiv 2(mod3)$

$$f(v_i^j) = \begin{cases} 2, if\ i \equiv 1(mod3) \\ 3, if\ i \equiv 2(mod3) \\ 1, if\ i \equiv 0(mod3) \end{cases} \tag{6}$$

For $j \equiv 0(mod3)$

$$f(v_i^j) = \begin{cases} 3, if\ i \equiv 1(mod3) \\ 1, if\ i \equiv 2(mod3) \\ 2, if\ i \equiv 0(mod3) \end{cases} \tag{7}$$

and

$$f(v_m^j) = \begin{cases} 2, if\ j \equiv 1(mod3) \\ 3, if\ j \equiv 2(mod3) \\ 1, if\ j \equiv 0(mod3) \end{cases} \tag{8}$$

The color classes $C[1]$, $C[2]$ and $C[3]$ satisfy the conditions $\left||C[i]|-|C[j]|\right| \leq 1, 1 \leq i \leq 3$ and $1 \leq j \leq 3$. The prism $Y_m^n$ admits the equitable coloring with this type of coloring.

Hence $\chi_=(Y_m^n) = 3$ if m is odd and $m \equiv 1(mod 3)$.

Therefore the chromatic number of $Y_m^n$ satisfies the inequality $2 \leq \chi_=(Y_m^n) \leq 3$.

**Illustration 1**

Consider the prism $Y_4^3$, where m is even. Using the above theorem case (i) we assign the color 1 for the vertices $v_1^1, v_3^1, v_2^2, v_4^2, v_1^3, v_3^3$ and the color 2 for the vertices $v_2^1, v_4^1, v_1^2, v_3^2, v_2^3, v_4^3$ as shown in figure 2.

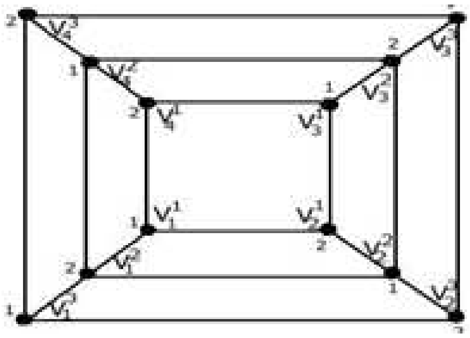

**Figure 2**

Here $|C[1]| = 6$, $|C[2]| = 6$ and these satisfy the condition $\left||C[1]|-|C[2]|\right| < 1$. This type of coloring the prism $Y_4^3$ satisfies the conditions for equitable coloring.

Hence $\chi_=(Y_4^3) = 2$.

**Illustration 2**

Consider the prism $Y_5^4$, where m is odd and $m \not\equiv 1(mod 3)$. Using the above theorem case (ii) we assign the color 1 for the vertices $v_1^1, v_4^1, v_3^2, v_2^3, v_5^3, v_1^4, v_4^4$, the color 2 for the vertices $v_2^1, v_5^1, v_1^2, v_4^2, v_3^3, v_2^4, v_5^4$ and the color 3 for the vertices $v_3^1, v_2^2, v_5^2, v_1^3, v_4^3, v_4^4$ as shown in figure 3.

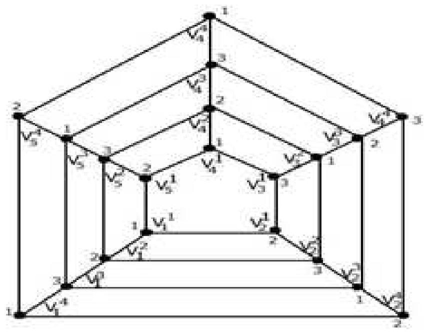

**Figure 3**

Here $|C[1]| = 7$, $|C[2]| = 7$ and $|C[3]| = 6$. these satisfy the conditions $\left||C[i]|-|C[j]|\right| \leq 1, 1 \leq i \leq 3$ and $1 \leq j \leq 3$. This type of coloring the prism $Y_5^4$ satisfies the conditions for equitable coloring.

Hence$\chi_=(Y_5^4) = 3$.

**Illustration 3**

Consider the prism $Y_7^4$, where m is odd and $m \equiv 1(mod3)$. Using the above theorem case (iii) we assign the color 1 for the vertices $v_1^1, v_4^1, v_3^2, v_6^2, v_2^3, v_5^3, v_7^3, v_1^4, v_4^4,$ the color 2 for the vertices $v_2^1, v_5^1, v_7^1, v_1^2, v_4^2, v_3^3, v_6^3, v_2^4, v_5^4, v_7^4$ and the color 3 for the vertices $v_3^1, v_6^1, v_2^2, v_5^2, v_7^2, v_1^3, v_4^3, v_3^4, v_6^4$ as shown in figure 4.

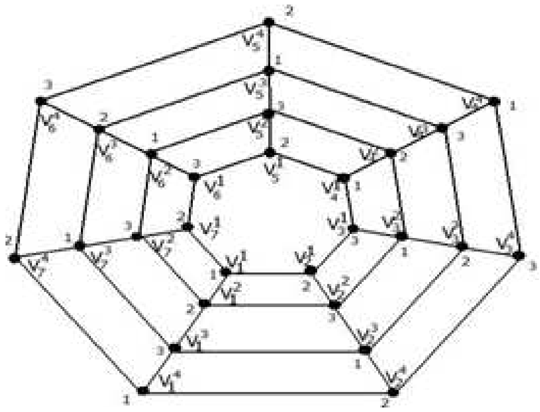

**Figure 4**

Here $|C[1]| = 9$, $|C[2]| = 10$ and $|C[3]| = 9$. They satisfy the conditions $\big||C[i]| - |C[j]|\big| \leq 1, 1 \leq i \leq 3$ and $1 \leq j \leq 3$ This type of coloring the prism $Y_5^4$ satisfies the conditions for equitable coloring.

Hence $\chi_{=}(Y_5^4) = 3$.

## EQUITABLE COLORING OF A GENERALIZED PETERSEN GRAPH $P(m,n), m \geq 2n+1, n > 1$

**Theorem 2**

The Petersen graph $P(2n, n)$ has the equitable chromatic number 2 for all $n \leq 3$.

**Proof:** The inner vertices of the Petersen graph $P(2n, n)$ are denoted by $u_1, u_2, u_3, \dots, u_n$ successively and vertices diametrically opposite to these vertices are denoted by $v_1, v_2, v_3, \dots, v_n$. The outer vertices adjacent to $u_1, u_2, u_3, \dots, u_n$ are denoted by $x_1, x_2, x_3, \dots, x_n$ ane the outer vertices adjacent to $v_1, v_2, v_3, \dots, v_n$ are denoted by $y_1, y_2, y_3, \dots, y_n$ as shown in figure 5.

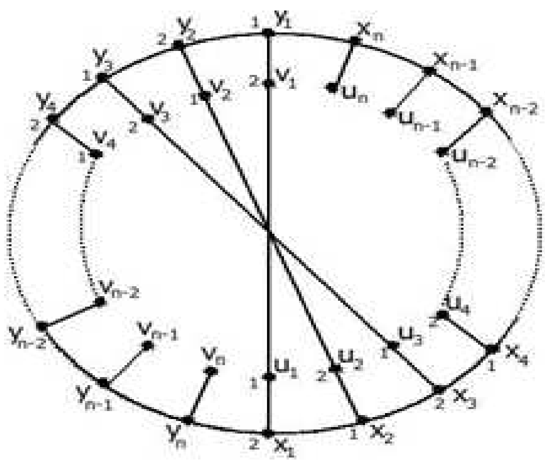

**Figure 5**

The function f is defined from the vertex set of $P(2n,n)$ to the set of colors $\{1,2,3,\dots\}$ for $1 \le i \le n$ as follows:

$$f(u_i) = \begin{cases} 1, if\ i\ is\ even \\ 2, if\ i\ is\ odd \end{cases} \tag{9}$$

$$f(v_i) = \begin{cases} 2, if\ i\ is\ even \\ 1, if\ i\ is\ odd \end{cases} \tag{10}$$

$$f(x_i) = \begin{cases} 1, if\ i\ is\ even \\ 2, if\ i\ is\ odd \end{cases} \tag{11}$$

$$f(y_i) = \begin{cases} 2, if\ i\ is\ even \\ 1, if\ i\ is\ odd \end{cases} \tag{12}$$

The color classes $C[1]$ and $C[2]$ satisfy the conditions $\big||C[1]| - |C[2]|\big| < 1$. The Petersen graph $P(2n,n)$ has the equitable coloring.

Hence $\chi_{=}(P(2n,n)) = 2$.

**Illutration 4**

Consider the Petersen graph $P(6,3)$, where $m = 2n$. By using theorem 2, we assign the color 1 to the vertices $u_1, u_3, v_2, x_2, y_1, y_3$ and the color 2 to the vertices $u_2, v_1, v_3, x_1, x_3, y_2$ as shown in the figure 6.

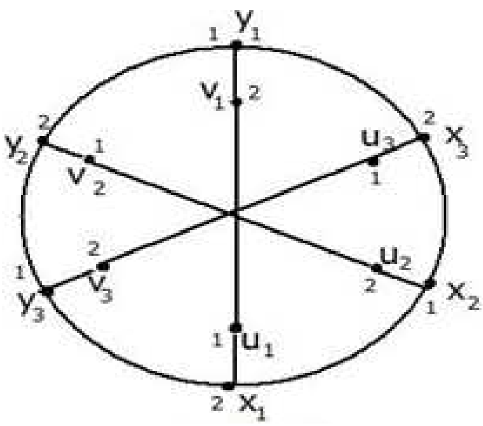


**Figure 6**

Here $|C[1]| = 6$, $|C[2]| = 6$ and they satisfy the conditions $\big||C[1]| - |C[2]|\big| < 1$. This type of coloring of the Petersen graph $P(6,3)$ satisfies the conditions for equitable coloring.

Hence $\chi_{=}(P(m,n)) = 2$.

**Theorem 3**

The Petersen graph $P(m,n)$ has the equitable chromatic number 3 if $\frac{m}{n} \equiv 0(mod 3)$.

**Proof:** If $\frac{m}{n} \equiv 0(mod3)$ then the inner vertices of Petersen graph $P(m,n)$ has n cycles of order 3, let the cycles be denoted by $u_1v_1w_1, u_2v_2w_2, \ldots, u_nv_nw_n$. We define the function $f$ from the set of inner vertices of $P(m,n)$ to the color set $\{1,2,3,\ldots\}$ for $1 \le i \le n$ as follows:

$f(u_i) = 1,$

$f(v_i) = 2,$

$f(w_i) = 3.$

The outer vertices adjacent to $u_i$'s are denoted by $x_i$'s; the outer vertices adjacent to $v_i$'s are denoted by $y_i$'s and the outer vertices adjacent to $w_i$'s are denoted by $z_i$'s as shown in figure 7.

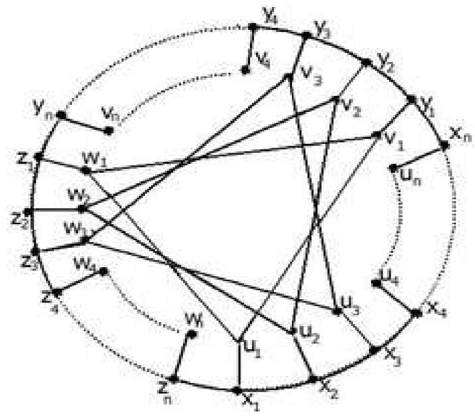

**Figure 7**

We define the function $f_1$ from the outer vertices of $P(m,n)$ to the color set $\{1,2,3,\ldots\}$ for $1 \le i \le n$ as follows:

$$f_1(x_i) = \begin{cases} 3, if\ i\ is\ odd \\ 2, if\ i\ is\ even \end{cases} \tag{13}$$

The outer vertices $x_i$'s adjacent to the inner vertices $u_i$'s are assigned with the colors either 2 or 3.

$$f_1(y_i) = \begin{cases} 1, if\ i\ is\ odd \\ 3, if\ i\ is\ even \end{cases} \tag{14}$$

The outer vertices $y_i$'s adjacent to the inner vertices $v_i$'s are assigned with the colors either 1 or 3.

$$f_1(z_i) = \begin{cases} 2, if\ i\ is\ odd \\ 1, if\ i\ is\ even \end{cases} \tag{15}$$

The outer vertices $z_i$'s adjacent to the inner vertices $w_i$'s are assigned with the colors either 1 or 2.

The color classes $C[1]$, $C[2]$ and $C[3]$ satisfy the conditions $\big||C[i]| - |C[j]|\big| \le 1, 1 \le i \le 3$ and $1 \le j \le 3$. The Petersen graph $P(m,n)$ has the equitable coloring.

Hence $\chi_{=}(P(m,n)) = 3$ if $\frac{m}{n} \equiv 0(mod3)$.

**Illutration 5**

Consider the Petersen graph $P(6,2)$, where $\frac{m}{n} \equiv 0(mod3)$. By using theorem 3, we assign the color 1 to the vertices $u_1, u_2, y_1, z_1$, the color 2 to the vertices $v_1, v_2, x_2, z_2$ and the color 3 to the vertices $w_1, w_2, x_1, z_2$ as shown in the figure 8.

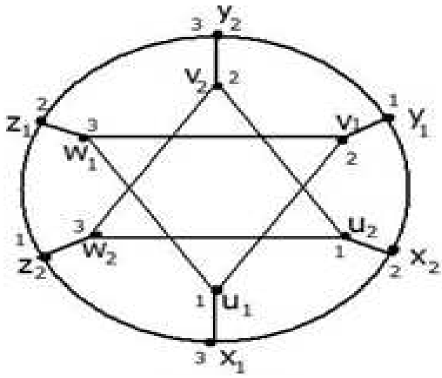


**Figure 8**

Here $|C[1]| = 4$, $|C[2]| = 4$ and $|C[3]| = 4$. It satisfy the condition $\big||C[i]| - |C[j]|\big| < 1, 1 \leq i \leq 3$ and $1 \leq i \leq 3$. This type of coloring the Petersen graph $P(6,2)$ satisfies the conditions of equitable coloring.

Hence $\chi_{=}(P(6,2)) = 3$.

**Theorem 4**

If $\frac{m}{n}$ is a positive integer then the equitable chromatic number of $P(m,n)$ satisfies the inequality $2 \leq \chi_{=}(P(m,n)) \leq 3$.

**Proof:** The generalized Petersen graph $P(m,n)$ for $m \geq 3$ and $1 \leq n \leq \left\lfloor\frac{m-1}{2}\right\rfloor$ is a graph consisting of an inner star polygon $\{m,n\}$ and an outer polygon $\{m\}$ with corresponding vertices in the innerand outer polygon connected with edges. $P(m,n)$ has 2m vertices. If $\frac{m}{n}$ is a positive integer, we assign the color from the set either $\{1,2\}$ or $\{1,2,3\}$ to the vertices of $P(m,n)$ in such away that no two adjacent vertices have the same color and the number of vertices in any two color classes differ atmost by one. Hence the Petersen graph $P(m,n)$ admits the equitable coloring and it takes either 2 or 3 colors and so the equitable chromatic number of $P(m,n)$ satisfies the inequality $2 \leq \chi_{=}(P(m,n)) \leq 3$.

## REFERENCES

1. Mayer. W, Equitable coloring, American Mathematical Monthy 80 (1973) P:143-149.
2. K. W. Lih, The Equitable coloring of graphs, handbook of combinatorial optimization, Vol-33, Kluwer Academic publishers, Boston MA, 1998.
3. D. Grittner, TheEquitable chromatic number for a complete bipartite graph, Undergraduate honors thesis, Millersville university PA, 1995.
4. Dorothee Blum, Equitable chromatic number of complete multipartite graphs, Millersville University, Millersville, PA 17551.
5. Coxter H. S. M.(1950), Self-dual configurations and regular graphs, Bulletin of the American Mathematical Society, 56, P: 413-455.